



Nationwide Inclusion and Exclusion Thresholds of Representation

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The theoretical inclusion and exclusion thresholds are, respectively, the vote shares below which a party cannot possibly win a seat, and above which it cannot possibly fail to do so. They are important in evaluating how hospitable electoral systems are to small parties. Previously, they have been calculated at the district level. Here the theory is extended to the national level. Surprisingly, the inclusion threshold depends on the smallest district in the country — not the largest. The exclusion threshold depends on all districts. The theoretical results are compared to empirical observations for 23 electoral systems. The inclusion threshold is indeed close to the minimal vote share that ever led to a seat in the national assembly. In stark contrast, the exclusion threshold is much higher than the maximal vote share that ever failed to produce a seat in practice. The total number of districts emerges as a significant variable. © 1998 Elsevier Science Ltd. All rights reserved

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Theoretical inclusion and exclusion thresholds have long fascinated students of electoral systems, starting at least with Rokkan (1968), because they open a potential avenue for systematically predicting the degree to which an electoral system restricts small-party access to representative assembly. However, the promise has long been stymied by inability to extend district-level threshold calculations to the nationwide level where the main interest lies.

The purpose of this study is to carry out such an extension from district to nationwide and to compare the theoretical results with actual data for stable democracies. It is found that the theoretical inclusion threshold does reflect the actual minimal vote share at which seats are won. In contrast, the theoretical exclusion threshold is much higher than the actual maximal vote shares at which parties still fail to win a seat in a national assembly. As a result, an average threshold of nationwide representation cannot be estimated directly from the inclusion and exclusion thresholds. It is found that it may be done, however, when the number of districts is also taken into account.

From District to Nationwide Thresholds

In most electoral systems seats are allocated within electoral districts. What is the minimum vote share with which a party can win at least one seat in the given district, under the luckiest possible conditions? This question defines T_I , the threshold of inclusion (or of representation, as Rokkan originally called it in 1968). T_I depends on the seat allocation formula. For a given PR (proportional representation) formula T_I decreases with increasing district magnitude (M , the number of seats allocated in the district) and also with increasing number of parties competing (n_v).

The complementary question is: What is the maximum vote share at which a party can still fail to win a seat in the given district, under the unluckiest conditions possible? This 'threshold of exclusion' (T_E) was first investigated by Rae *et al.* (1971). It, too, depends on the allocation formula and decreases with increasing M . However, it does not depend on n_v , because the unluckiest condition always occurs with $n_v = 2$, which means facing a united opposition. The actual winning of the first seat can occur anywhere in the range T_I to T_E , and a suitably defined intermediary value could be called the 'average threshold of representation'.

Inclusion and exclusion thresholds were most intensively investigated in the 1970s (Rae *et al.*, 1971; Loosemore and Hanby, 1971; Rae, 1971; Grofman, 1975; Lijphart and Gibberd, 1977; Laakso, 1979a, b), but later the activities tapered off (Lijphart, 1986, 1994; Taagepera and Shugart, 1989; Grofman, 1997), because the district-level investigation was completed, while extension to the nationwide did not take off.¹

At the hard analytical level the nationwide extension seemed to involve unmanageable parameters, such as geographical distribution of party strengths. At a softer level this analytical inability was tucked under the carpet: It was tacitly assumed that district thresholds would be a fair approximation for the nationwide, despite contrary evidence that was especially strong for $M = 1$ (Taagepera, 1989, 1998). Indeed, while average thresholds ranging from 35 to 50% have been proposed in individual single-member districts (Lijphart, 1994; Taagepera and Shugart, 1989), seats in the UK parliament were actually won with 0.3% national votes and even less. This incongruity motivated the present attempt to investigate the nationwide thresholds, and the solution was unexpectedly straightforward.

In this study I first calculate the nationwide inclusion threshold (T_I) as percentage of the national vote. This theoretical boundary is compared to the empirically determined lowest vote shares at which the first seat actually has been won (v_I). The procedure is repeated for the nationwide exclusion threshold (T_E). The highest empirical percentage at which a party still has been observed to fail to win a seat in the assembly is designated as v_E .²

The theoretical part of this study deals with the d'Hondt allocation rule only. Not only is it the most frequently used, among the list PR rules, but it also offers the highest inclusion and the lowest exclusion thresholds.³ The actual data include systems that use other allocation rules, too, thus putting the theoretical results to the severest possible test.

If the seats in an electoral district are allocated according to the d'Hondt divisors (1, 2, 3,...), then the district-level inclusion threshold is

$$T_I' = 100\%/(M + n_v - 1), \quad (1)$$

and the exclusion threshold is

$$T_E' = 100\%/(M + 1), \quad (2)$$

independent of the number of parties competing (n_v).⁴ The apostrophes are used here and later to distinguish district-level thresholds from the nationwide.

Nationwide Theory

Inclusion Threshold

The nationwide inclusion threshold T_I corresponds to the situation where a party has all its votes concentrated in one single district, where it barely carries one seat. If district magnitudes within the country vary (as they most often do), can a seat be won with fewer nationwide votes in a small or a large district? It may seem obvious that the largest M gives a small party its best chance, because this is the case at district level (cf. Equation (1)). Surprisingly, however, the reverse is true at the national level: It's in the district with the smallest M that the least nationwide votes are needed to win a seat. Such a reversal in the effect of M at district and national levels has been previously noted by Grofman (1997). The proof is as follows.

If the same number of parties is running in each district (an assumption soon relaxed), the district-level threshold T_I' is, indeed, the lowest in the district with the largest M , according to Equation (1). However, a party's nationwide vote share is a fraction M/S of the district share (S being the number of seats in the national assembly):

$$T_I = T_I' M / S = (100\% / S) [M / (M + n_v - 1)]. \quad (3)$$

At constant S , T_I is minimized for the smallest M . Hence the nationwide votes that still can win a seat are minimized when all the votes are concentrated in the lowest-magnitude district in the country. The national inclusion threshold is, therefore,

$$T_I = (100\% / S) M_m / (M_m + n_v - 1), \quad (4)$$

where M_m stands for minimum M that occurs.⁵

Now the condition that n_v be the same in all districts is relaxed, as it should, because more parties can win seats in a larger district. Hence, when a country has districts of varying M , the larger districts may see more parties running. The approximation $n_v = M^{.5} + 1$ is proposed here, based on the following reasoning. Other possible approximations for n_v are discussed in the Appendix.

The number of seat-winning parties (n_s) in a district is restricted by M : It must be at least 1 and can be at most M . An actual observation is that for $M = 100$ in the Netherlands 1918–52 (a single nationwide district), $n_s = 10$, which is square root of M . More generally, the relationship $n_s = M^{.5}$ has been proposed on broader probabilistic grounds (Taagepera and Shugart, 1993). The number of competing parties can be appreciably larger than the number of seat-winning parties, but in stable electoral systems the number of serious contenders tends to be only slightly above n_s . We may tentatively assume $n_v = n_s + 1$. This is in line with Reed (1991), who observed $M + 1$ candidates running in Japan, where each candidate competed with all the other candidates, even of the same party. Combined with the previous, it leads to $n_v = M^{.5} + 1$. Now, at the district level, Equation (1) becomes

$$T_I' = 100\% / (M + M^{.5}), \quad (5)$$

and, at the national level, Equation (3) becomes

$$T_I = (100\%/S)/(1 + M^{-.5}). \quad (6)$$

The nationwide threshold decreases with decreasing M and is minimized when the smallest of the district magnitudes is chosen. Equation (6) becomes

$$T_I = (100\%/S)/(1 + M_m^{-.5}). \quad (7)$$

Exclusion Threshold

In a single district the inclusion and exclusion thresholds play a symmetrical role in the sense that winning the seat by the least possible margin and failing to do so by the least possible margin are both equally likely (or rather, unlikely). Moreover, both thresholds are fairly close to each other; the ratio ranges from 1 to 1.21 at most (if Equation (5) holds).⁶ Hence the average threshold of representation can be set in the middle of the zone of possible outcomes. Surprisingly, this symmetry breaks down at the national level, where the gap between exclusion and inclusion thresholds widens markedly.

The nationwide inclusion threshold involves one party or independent running only in one small district and narrowly winning a seat. This is a situation that happens, though rarely. In contrast, the nationwide exclusion threshold corresponds to the situation where a party runs in every district and narrowly fails to win a seat in every single one of them. This is much less likely, because it compounds low probabilities in each district.

Therefore, while some actual wins might be just slightly above the theoretical T_I , one would expect even the highest vote shares that fail to win a seat to be lower than T_E by a wide margin. Hence the nationwide average threshold of representation is expected to be lower than the mean of the inclusion and exclusion thresholds.

The nationwide exclusion threshold is the sum of all district thresholds, $T_E' = 100\%/(M_i + 1)$, each weighted by the given district's share (M_i/S) of the national vote:

$$T_E = (100\%/S)\Sigma M_i/(M_i + 1). \quad (8)$$

Sometimes we do not have detailed information on the distribution of magnitudes in the country but only the total number of seats and the number of districts (E). In such a case the average magnitude ($M_{av} = S/E$) can be used to obtain an upper limit on the exclusion threshold:

$$T_E \leq 100\%/(1 + M_{av}). \quad (9)$$

The equality is valid only when all districts have the same magnitude. In most actual cases the actual T_E is below this upper limit by only 1% point. The largest shortfalls (up to 8% points) occur when one-third to one-half of the seats are allocated in one-seat districts and the rest in one huge one.⁷

The Ratio of Exclusion and Inclusion Thresholds

The expression for the ratio $r = T_E/T_I$ resulting from Equation (7) and Equation (9) is complex and confusing. However, when the number of districts (E) is introduced, the picture is simplified: the product rE varies only between 1 and 1.21 — the same range that applies to the ratio of exclusion and inclusion thresholds in a single district. In a very broad sense, this is understandable, given that the inclusion threshold depends on only one district, while the exclusion threshold depends on all E districts. But the narrowness of the available zone (1.00–1.21) is surprising, because these districts may be very small (even $M = 1$) or huge. Yet the contrast between the nationwide exclusion and inclusion thresholds widens about proportionately to the number of districts, little affected by average district magnitude and hence total assembly size.

Up to now, electoral systems theory has paid considerable attention to the magnitude of districts but has neglected the impact of how many districts there are. Here the number of districts emerges as a variable of some importance.

Comparison with Observed Values

The comparison includes all periods for all countries listed in Mackie and Rose (1991) where seat allocation took place purely in districts and electoral rules remained the same for at least three elections. Inclusion required that information on electoral rules and the number of districts could be determined reasonably well, based mainly on Nohlen (1978). The long period for the United States (1828–1988) was broken up into three, the last of which (1938–88) had to be omitted because there were no third parties narrowly winning or failing to win seats. Some other periods in other countries were omitted for the same reason. Of the 23 systems thus determined, seven used multiseat d'Hondt, and six used its $M = 1$ equivalent (plurality). The other ten cases include three two-rounds systems, four Single Transferable Vote and Alternative Vote systems, two with Sainte-Laguë allocation rule and one Single Non-Transferable Vote. For each system the lowest vote share that actually won a seat (v_I) was determined, and so was the highest vote share that failed to win a seat (v_E).

In all the cases the d'Hondt-based theoretical formulas were used. Some of the other systems are more hospitable to small parties: their theoretical district-level inclusion thresholds are somewhat lower and exclusion thresholds somewhat higher than for d'Hondt. Thus the test is two-fold: are the theoretical expressions confirmed by data on the d'Hondt systems? and are these formulas sufficiently robust to be extended to other, more liberal systems?

Inclusion Threshold

Table 1 shows the countries and periods listed in the decreasing order of the number of electoral districts (E). Shown are the average magnitude (M_{av}), minimum magnitude (M_m), inclusion threshold T_I calculated from Equation (7), and the observed v_I . There is some confusion regarding the number of districts in Sweden 1908–48. The median v_I/T_I ratio is 1.8, confirming that the actual minimal nationwide votes leading to a seat tend to be not much higher than the theoretical limit.

Fig. 1 shows v_I graphed against T_I , both on logarithmic scale so as to show more detail at low values. The equality line $v_I = T_I$ is also shown. Different symbols are used for different allocation rules. The points tend to crowd moderately above the equality line (the exceptions

Table 1. Calculated nationwide inclusion thresholds and the observed minimal winning votes

Country, period, and no. of elections	No. of distr. ^a E	Aver. magn. M_{av}	Min. magn. M_m	Inclusion threshold ^b T_1 (%)	Observed min. vote ^c v_1 (%)
UK 1922–87, 19	628.2	1	1	0.08	0.13
* France 1958–81, 7	470.1	1	1	0.11	1.5
U.S. 1884–1936, 27	396.2	1	1	0.13	0.4
* Germany 1871–1912, 13	395.8	1	1	0.13	0.14
Canada 1878–1988, 31	246.9	1	1	0.20	0.7
U.S. 1828–82, 28	239.9	1	1	0.21	1.5
* Japan 1928–86, 22	120	4.0	1	0.10	0.4
Norway 1882–1903, 8	114.4	1	1	0.44	9.7
* Australia 1919–87, 28	106.0	1	1	0.47	1.0
* Netherl. 1888–1913, 8	100	1?	1	0.50	0.9
N. Zealand 1890–1987, 32	80.8	1	1	0.62	1.0
Spain 1977–86, 3	52	6.7	1	0.14	0.2
* Ireland 1922–89, 24	42.7	3.5	3	0.42	0.21
Sweden 1908–48, 14	56/28	4.1/8.2	3	0.28	0.17
* Sweden 1952–68, 6	26	8.9	2	0.25	0.32
Switzerland 1919–87, 19	25	7.8	1	0.25	0.4
Portugal 1975–87, 7	22	11.3	1	0.20	0.8
* Norway 1953–85, 9	19.5	7.8	3?	0.42?	2.3
Norway 1921–49, 8	20	7.5	3	0.42	0.55
Finland 1907–87, 30	15	13.3	1	0.25	0.4
* Malta 1947–87, 11	10.0	5	5	1.38	2.6
* Malta 1921–45, 6	6.2	4	4	2.70	8.8
Luxembourg 1919–89, 11	4	13.7	6	1.29	1.6

*Using an allocation rule other than d'Hondt or $M = 1$ plurality.

^aThe number of electoral districts (E), average district magnitude (M_{av}), minimum district magnitude (M_m): mainly based on Nohlen (1978) and Lijphart (1994).

^bInclusion threshold calculated from $T_1 = (100\%/S)/(1 + M_m^{-5})$, where $S = EM_{av}$.

^cObserved minimal nationwide votes that won a seat (v_1), from data in Mackie and Rose (1991).

will be discussed shortly). For one-half the cases the observed v_1/T_1 ratio is between 1.0 and 2.0. Thus the calculated inclusion threshold appears indeed as the lower limit at which seats can be won. Countries that use another allocation rule than d'Hondt do not stand out in a systematic way, although the theoretical values are based on d'Hondt specifically.

Two points fall below the equality line. In Ireland (STV rule) the lowest-vote victory was the Green Party's in November 1982. Note that the v_1 value used is based on the first-preference votes only; transfer votes can raise its value appreciably. The other STV and AV systems (Australia and two periods for Malta) have even the first-preference v_1 above the calculated T_1 . The other exception is Sweden 1908–48, where minor candidates (unspecified in Mackie and Rose, 1991) won not one but two seats with a mere 0.17% votes in 1920 — a feat explained by apparentement (Cox, 1997, p. 61).

The unusually high points (such as Norway 1882–1903 and France 1958–81) need no explanation, given that small parties just above the inclusion threshold need not materialize. Indeed, it is remarkable that in so many cases they did. One might be attempted to connect France's high value of v_1 to its use of two-rounds elections, but Imperial Germany used the same system and produced some winning parties barely above the theoretical threshold.

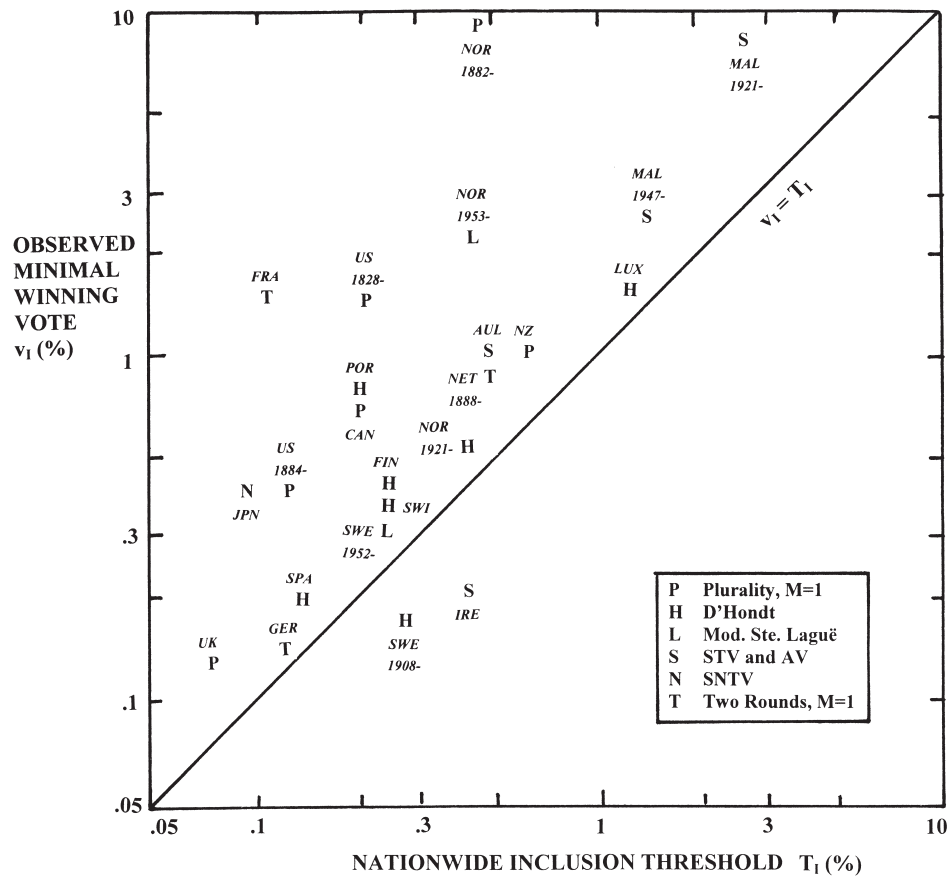


Fig. 1. The lowest nationwide seat-winning vote share observed (v_1) vs the theoretical nationwide inclusion threshold (T_1).

Exclusion Threshold and the Number of Electoral Districts

Table 2 shows the calculated exclusion thresholds (T_E) and the observed maximal failing votes (v_E) for the same electoral systems as in Table 1. The data on E and M_{av} is repeated for comparison purposes. Disregard for the moment the last column (T_E/E^5).

Fig. 2 shows observed v_E graphed against the calculated T_E , using the same format as in Fig. 1. As expected, all the observed v_E fall below the theoretical threshold by appreciable margins. The median T_E/v_E ratio is 7.1 (as compared to the median v_1/T_1 ratio of only 1.8). For countries with relatively few electoral districts ($E < 70$) the median ratio is moderate (3.2), but the gap becomes huge (median ratio 11) in the case of systems with very numerous districts ($E > 70$), meaning single-member systems and Japan. Narrowly failing in each of these numerous districts, as required for T_E , would indeed defy the laws of probability.⁸

The threshold of exclusion could be reached only if a party ran in all districts and narrowly lost in each of them. Actually, a small party rarely runs everywhere. The number of districts in which it can run ranges in principle from 1 to E . In the absence of any other information the most rational guess would be the geometric mean of these extremes. In such a case the theoreti-

Table 2. Calculated nationwide exclusion thresholds and the observed maximal failing votes

Country and period	No. of distr. E	Average magn. M_{av}	Exclusion threshold ^a T_E (%)	Observed max. ^b v_E (%)	Ratio $T_E/E^{.5}$ (%)
UK 1922–87	628.2	1	50.0	0.6	2.0
* France 1958–81	470.1	1	50.0	3.9	2.4
U.S. 1884–1936	396.2	1	50.0	6.4	2.5
* Germany 1871–1912	395.8	1	50.0	1.2	2.5
Canada 1878–1988	246.9	1	50.0	4.6	3.2
U.S. 1828–82	239.9	1	50.0	2.2	3.2
* Japan 1928–86	120	4.0	20.0	2.5	1.8
Norway 1882–1903	114.4	1	50.0	3.0	4.7
* Australia 1919–87	106.0	1	50.0	9.4	4.8
* Netherl. 1888–1913	100	1?	50.0	6.7	5.0
N. Zealand 1890–1987	80.8	1	50.0	12.3	5.6
Spain 1977–86	52	6.7	12.3	4.2	1.7
* Ireland 1922–89	42.7	3.5	22 –	3.1	3.4
Sweden 1908–48	56/28	4.1/8.2	20/10	0.9	2.6/1.9
* Sweden 1952–68	26	8.9	11 –	1.8	2.2
Switzerland 1919–87	25	7.8	9.8	1.3	2.0
Portugal 1975–87	22	11.3	7.6	1.4	1.6
* Norway 1953–85	19.5	7.8	11 –	3.4	2.5
Norway 1921–49	20	7.5	13 –	5.8	2.9
Finland 1907–87	15	13.3	6.8	2.2	1.8
* Malta 1947–87	10.0	5	16.7	6.0	5.3
* Malta 1921–45	6.2	4	20.0	11.0	8.0
Luxembourg 1919–89	4	13.7	6.2	2.3	3.1

*Using an allocation rule other than d'Hondt or $M = 1$ plurality.

^aExclusion threshold: with decimals — calculated from $T_E = (100\%/S)\Sigma SM_i/(M_i + 1)$, where M_i comes mainly from Nohlen (1978); with a minus sign — upper estimate from $T_E = 100\%/(M_{av} + 1)$. M_{av} from Table 1.

^bObserved maximum nationwide votes that failed to win a seat (v_E), from data in Mackie and Rose (1991).

cal T_E would be divided by $E^{.5}$. This quantity is also shown in Table 2. Now a more systematic pattern emerges. Within a factor of 4, the following applies:

$$v_E = T_E/E^{.5}. \quad (10)$$

Of the observed v_E values 13 are above and 10 below this level. The outcome does not seem to depend on the type of electoral rules used, apart from the number of districts.⁹

Average Threshold of Nationwide Representation

At the district level the mean of inclusion and exclusion thresholds gives a fair idea of the vote share at which parties have a 50–50 chance of winning a seat. At the nationwide level the excessive nature of T_E makes the mean of T_E and T_I unsuitable for this purpose: instead of being in between the actual v_I and v_E the geometric mean is higher than the observed v_E in one-third of the cases — and the arithmetic mean is even higher. Taking the geometric

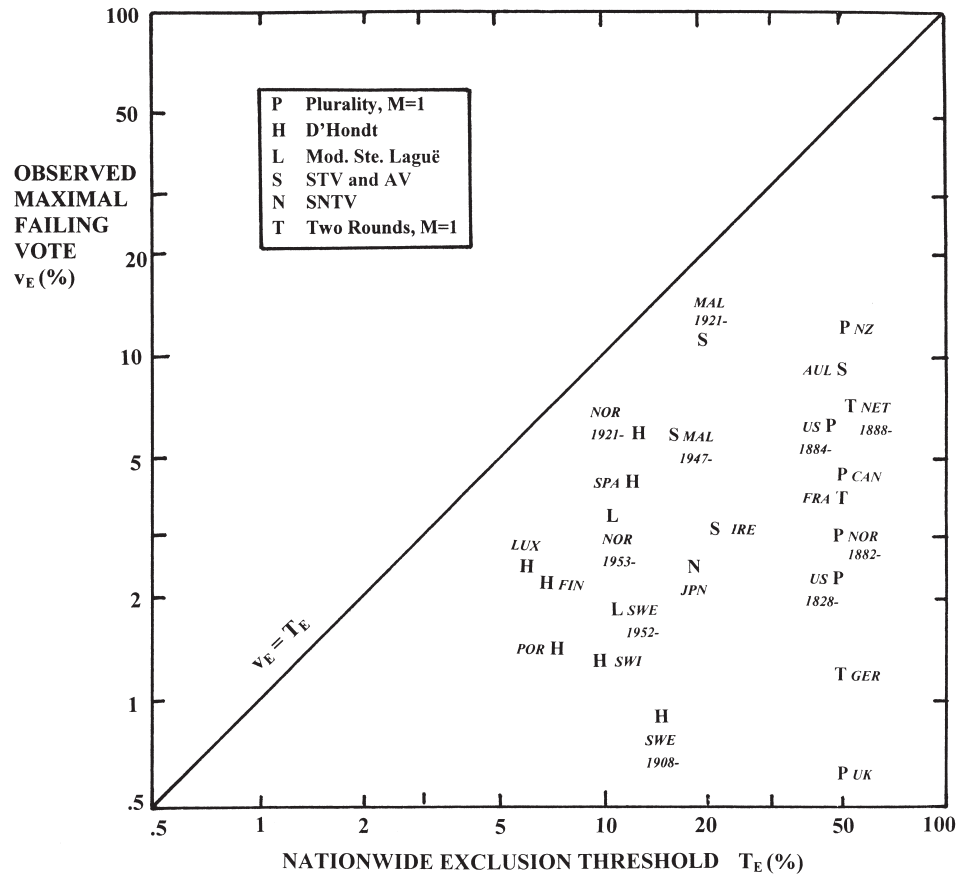


Fig. 2. The highest nationwide failing vote share observed (v_E) vs the theoretical nationwide exclusion threshold (T_E).

mean of $T_E E^{.5}$ and T_I corrects excessively: now one-quarter of the presumed averages are actually below the observed lower threshold, v_I . Finally, dividing the geometric mean of T_I and T_E by $E^{1/8}$ yields a value that is in between the observed v_I and v_E in all cases except UK. Thus the average threshold of nationwide representation could be estimated as

$$T_{av} = (T_I T_E)^{.5} / E^{.125}. \quad (11)$$

The precise power of E used is empirical. However, the broad idea of dividing by some power of E expresses the notion that the more districts there are the lesser the likelihood that a party would run and narrowly fail in all of them.

Conclusion

What has been achieved? Intellectually, the longstanding quest for theoretical inclusion and exclusion thresholds has been brought to a logical conclusion by reaching the national level. The results include some surprises. The ways to calculate the inclusion and exclusion thresholds diverge, and the symmetry they offer at the district level breaks down at the national level.

The nationwide inclusion threshold has some reality in that numerous electoral systems have indeed generated outcomes just above the theoretical threshold. The surprising part is that the nationwide inclusion threshold depends on the magnitude of the smallest district, not the largest.

The nationwide exclusion threshold depends on all the districts. The surprise here is that it is so high that it is a poor guide for estimating the maximal vote share at which actual failures to win a seat occur. This is unlike the nationwide inclusion threshold, and also unlike the exclusion threshold at the district level.

The number of electoral districts emerges as an important intervening variable. One can estimate the actual highest vote share at which parties fail to win a seat in the national assembly, by dividing the theoretical exclusion threshold by the square root of the number of districts.

One can also estimate the average threshold of representation, i.e. the vote share at which a party has a 50–50 chance of winning a seat in the national assembly. This is done by taking the geometric mean of the theoretical inclusion and exclusion thresholds and then dividing by the eighth root of the number of districts. The latter correction is empirical, with some rational justification.

The specific theoretical calculations used here are based on the d'Hondt allocation rule. The empirical comparison included electoral systems using a variety of allocation rules, and no systematic differences were noticed. Similarly, the outcomes seem quite robust against different assumptions made regarding the number of parties running.

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Notes

1. Rae *et al.* (1971) extended Rokkan's (1968) analysis to exclusion thresholds. Loosemore and Hanby (1971) also calculated the maximum deviation from PR that could result. Rae (1971, p. 193) gave formulas for vote shares needed to win any number of seats (not just the first one). This work dealt with the d'Hondt, Sainte-Laguë and Largest Remainder allocation rules. Grofman (1975) extended the calculations to Bloc Vote, Cumulative Vote, Limited Vote and Modified Ste.-Laguë. Lijphart and Gibberd (1977) published detailed proofs, also correcting some previous errors. Laakso (1979a and 1979b) extended the formulas to some further allocation rules, focused on the threshold for winning all the seats, and compared deviations from PR. Lijphart (1986) corrected some further errors. Taagepera and Shugart (1989) offered an average threshold of representation that approximates the outcomes for most usual allocation formulas, and Lijphart (1994) refined it. All calculations and estimates were at district level.
2. Logically, the observed minimal seat-winning vote share must be larger than the inclusion threshold: $v_I > T_I$. It could be almost equal or appreciably higher than the theoretical lower boundary. Similarly, the observed maximal seat-failing vote share must be smaller than the exclusion threshold: $v_E < T_E$. It could be almost equal or appreciably lower than the theoretical upper boundary. In sum, we always have $T_I < T_E$, $v_I > T_I$, and $v_E < T_E$ — but these are the only logical constraints. Most often $v_I < v_E$, but the reverse can happen. This is the case for Norway 1882–1903 in Table 1 and Table 2.

3. For an overview of various seat allocation rules see e.g. Taagepera and Shugart (1989) or Lijphart (1994). At $M = 1$ the d'Hondt rule (like most list PR rules) boils down to plurality rule. Hence the thresholds calculated for d'Hondt apply to $M = 1$ plurality systems.
4. For instance, with $M = 3$ and $n_v = 4$ parties running, a party may land a seat with as little as a notch above $T_1 = 1/6 = 16.7\%$, if votes are distributed as $1/6+$, $1/6-$, $1/3-$, $1/3-$ (or $1/6+$, $1/6-$, $1/6-$, $1/2-$). On the other hand, it could still fail with slightly below $T_E = 1/4 = 25\%$, if votes are distributed as $1/4-$, $3/4+$ (or $1/4-$, $1/4+$, $1/4+$, $1/4+$).
5. The relationship can be subverted by malapportionment or turnout differentials in a direction that raises the simple quota in the smallest district sufficiently above the national average. Note that $100\%/S$ is the nationwide simple (Hare) quota.
6. For d'Hondt rule $T_E' - T_1'$ is largest around $M = 2.5$, where it amounts to 8.6% ($T_E = 28.6\%$, $T_1 = 20.0\%$). If $n_v = M^5 + 1$ holds, the ratio $r' = T_E'/T_1'$ is 1 for $M = 1$ and M tending toward infinity, reaching meanwhile a peak value of 1.207 around $M = 6$.
7. The summation in Equation (8) has E terms, and each of them can range only from 0.5 (for $M = 1$) to 1.0 (for extremely large M). Hence approximating all M_i by their mean, $M_{av} = S/E$, leads to a result that is rarely off by more than 1% for randomly distributed district magnitudes. Exceptions occur when the district magnitudes are sharply polarized: many one-seat districts and one huge district. The gap is the widest when 41.4% of the seats are in one-seat districts. Then, at very large S , the actual $T_E = 20.7\%$ (Equation (8)), while Equation (9) yields 29.3.
8. The ratio of the observed maximal failing and minimal winning votes for the same country might also be of interest. The median v_E/v_1 ratio is 5.6. It is largest for Spain (21) and smallest for Norway 1882–1903 (0.31). The latter result may look counterintuitive but is not, as long as both v_E and v_1 remain in the zone between the theoretical exclusion and inclusion thresholds — cf. Note 2.
9. Given that the nationwide threshold of inclusion involves the magnitude of the smallest district in the country, couldn't the nationwide threshold of exclusion be tied to the magnitude of the largest district (M_M)? If a very small party wanted to play it safe against exclusion, it would concentrate its resources in the largest district, where the district-level exclusion threshold is the lowest, and would not run candidates elsewhere. Such strategies occur in countries where one or two districts (typically in or around the capital city) are much larger than all others (Austria, Finland, Spain). This approach predicts a nationwide exclusion threshold of $T_E = (M_M/S)100\%/(M_M + 1)$. Unfortunately, much higher values of v_E are observed, especially when $M = 1$. Indeed, when all districts are equal, there is little reason for a small party to run in only one of them.

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Appendix

The Number of Parties Competing in a District

In order to calculate the inclusion threshold some assumptions had to be made regarding the relationship between district magnitude and n_v , the number of 'serious' parties and independents contesting the election. This is a separate problem of considerable importance in rational modeling of elections, and it is far from being resolved. Here a broad approach is outlined.

A possible general format for dependence of n_v on district magnitude M is $n_v = kM^A + 1$, where k and A are positive constants. Then the generalized analogue of Equation (3) is

$$T_I = T_I' M / S = (100\% / S) / (1 + kM^{(A-1)}).$$

The value $k = 1$ has some appeal, because at $M = 1$ it results in $n_v = 2$ — pure two-party competition. In practice, more than two competitors are frequently observed in single-member districts, but also some unopposed candidates. The value of A has more serious implications.

If we had $A > 1$, then T_I' would be smallest for the largest M , meaning that the nationwide inclusion threshold would materialize in the largest district. No one has proposed such a model. If, on the contrary, $A < 1$, then the nationwide T_I would materialize in the smallest district. This is the case for the model $n_v = M^{-5} + 1$ used in this study.

The critical point is $A = 1$. Then $n_v = kM + 1$, and M vanishes from the equation above. Regardless of M , $T_I = 100\% / S(1 + k)$, so that the seat won with minimal nationwide votes could occur in any district, small or large. As an upper limit, $n_v = M + 1$ has been proposed by Cox (1997), who has argued that at most $M + 1$ serious parties can be expected to run. It harks back to Reed's (1991) proposal, inspired by Japanese data, that the number of serious candidates (rather than parties) is $M + 1$. At this upper limit Equation (1) and Equation (3) yield, at district level, $T_I' = 50\% / M$ and, at national level, $T_I = 50\% / S$.

Under this assumption district thresholds would depend on district magnitude, but the national threshold would depend only on the total number of seats in the assembly. It would not matter whether the country is divided into many single-member districts or fewer multiseat districts of equal or unequal magnitude — the inclusion threshold would be the same. If there

are larger and smaller districts, it would not matter where the small party concentrates its votes: If lucky, it could win the single seat in a single-member district with exactly the same share of national vote as it would need to win one of the many seats in a large district.

It is time for some reality check. At $M = 1$ the models $n_v = M + 1$ and $n_v = M^{-5} + 1$ agree: Candidates of two parties will run. In small multiseat districts both models still give fairly similar predictions. Divergences occur at large M . Take the aforementioned Netherlands 1918–52, with $M = 100$. Reed's (1991) argument for $M + 1$ candidates probably should not be extended beyond SNTV, because 101 candidates appears much too low for the Netherlands. Several major parties may run a full slate, raising the number of candidates to several hundred. Cox's (1997) upper limit of $M + 1 = 101$ parties seems too high: given that only around 10 parties and independents are observed to win seats, it is hard to envisage as many as 101 serious (or even semiserious) parties and independent candidates run. On the other hand, $M^{-5} + 1 = 11$ parties and independents running, as predicted by the model used in the present study, appears too low.

These hunches are reinforced by a quick test at an intermediary magnitude. In Finland the median district magnitude is $M = 14$ (when omitting the lone single-member district in the Åland Islands) and the observed median district n_v is 8.0 (Taagepera and Shugart, 1989, p. 113). In comparison, $M + 1 = 15$ is much too high, while $M^{-5} + 1 = 4.7$ is much too low. The intermediary power index 0.75 comes closest: $M^{.75} + 1 = 8.2$. In this case Equation (8) should be replaced by

$$T_I = 100\%/S(1 + M_m^{-.25}).$$

Note that the nationwide T_I still depends on the smallest district. This spot check deals with only one country, but it raises the possibility that actual cases may fall in between $n_v = M + 1$ and $n_v = M^{-5} + 1$.

In the present study the latter model has been used, because it produces the highest value for T_I and hence puts the approach to the severest test: If T_I is overestimated, then it could happen that the observed v_I would fall below the predicted T_I , a logical impossibility. The present study finds that $v_I > T_I$ for all systems except two — and for these two the discrepancy would remain for $n_v = M + 1$ too.